Recall: Integration by Parts

$$\boxed{\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx}$$

How it works:

Given the integral of a product.

1) Choose one function to be $g'(x)$ - a function you can integrate!

2) what's left over is $f(x)$.

Then use the formula.

Integration by Parts Shorthand

$$f(x) = u, \quad dv = g'(x)\,dx$$

Then integration by parts becomes

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Choose dv = something you can integrate

Example 1: (integral of ln(x))

$\int \ln(x)\,dx$ using integration by parts

Cheap product: $\ln(x) = 1 \cdot \ln(x)$

$\boxed{u = \ln(x)}$ $\quad v = x$

$du = \frac{1}{x}\,dx$ $\quad \boxed{dv = 1\,dx}$

$$\boxed{\int u\,dv} = uv - \int v\,du$$

$u = \ln(x)$ $\quad$ $v = x$

$du = \frac{1}{x}\,dx$ $\quad$ $dv = 1 \cdot dx$

$$\boxed{\int \ln(x)\,dx} = x\ln(x) - \int \cancel{x} \cdot \frac{1}{\cancel{x}}\,dx$$

$$= x\ln(x) - \int 1\,dx$$

$$= \boxed{x\ln(x) - x + C}$$

Also works for definite integrals:

$$\boxed{\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x)\,dx}$$

Example 2:

$$\int_0^{\ln(2)} x e^x \, dx.$$ Try

$$\boxed{u = e^x} \qquad v = \frac{x^2}{2}$$

$$du = e^x \, dx \qquad \boxed{dv = x \, dx}$$

So by integration by parts,

$$\int_0^{\ln(2)} x e^x \, dx = \frac{x^2}{2} e^x \Big|_0^{\ln(2)} - \int_0^{\ln(2)} \frac{x^2}{2} e^x \, dx$$

Harder than the original integral!

Instead, try

$\boxed{u = x}$ $\quad$ $v = e^x$

$du = 1 \cdot dx$ $\quad$ $\boxed{dv = e^x\,dx}$

Using integration by parts,

$$\int u\,dv = uv - \int v\,du, \text{ so}$$

$$\int_0^{\ln(2)} x e^x\,dx = x e^x \Big|_0^{\ln(2)} - \int_0^{\ln(2)} e^x\,dx$$

$$= x e^x \Big|_0^{\ln(2)} - e^x \Big|_0^{\ln(2)}$$

$$xe^x\Big|_0^{\ln(2)} - e^x\Big|_0^{\ln(2)}$$

$$= \ln(2)\, e^{\ln(2)} - \left(e^{\ln(2)} - 1\right)$$

$$= \boxed{2\ln(2) - 1} \quad \text{since } e^{\ln(2)} = 2.$$

Example 3: (tabular)

$$\int x^5 \sin(2x)\,dx$$

$\boxed{u = x^5}$ $\qquad v = -\dfrac{\cos(2x)}{2}$

$du = 5x^4\,dx$ $\qquad \boxed{dv = \sin(2x)\,dx}$

So $\int x^5 \sin(2x)\,dx$

$$= -\frac{x^5\cos(2x)}{2} + \int \frac{\cos(2x)}{2} \cdot 5x^4\,dx$$

another integration by parts!

Trick: Tabular Method

Make a table

u		dv
x^5	$+$	$\sin(2x)$
$5x^4$	$-$	$-\cos(2x)/2$
$20x^3$	$+$	$-\sin(2x)/4$
$60x^2$	$-$	$\cos(2x)/8$
$120x$	$+$	$\sin(2x)/16$
120	$-$	$-\cos(2x)/32$
0		$-\sin(2x)/64$

1) Differentiate the u-column until you get zero.

2) Integrate dv as many times as you differentiated u.

3) Multiply diagonally, with negative signs on the even diagonals

4) Add all the terms.

That's the answer.

u		dv
x^5	$+$	$\sin(2x)$
$5x^4$	$-$	$-\cos(2x)/2$
$20x^3$	$+$	$-\sin(2x)/4$
$60x^2$	$-$	$\cos(2x)/8$
$120x$	$+$	$\sin(2x)/16$
120	$-$	$-\cos(2x)/32$
0		$-\sin(2x)/64$

Answer:

$$-\frac{x^5\cos(2x)}{2} - \frac{5x^4\sin(2x)}{4} + \frac{20x^3\cos(2x)}{8} - \frac{60x^2\sin(2x)}{16} - \frac{120x\cos(2x)}{32} + \frac{120\sin(2x)}{64} + C \; !$$

You can (and should) use the tabular method for

1) (polynomial) · (exponential)

2) (polynomial) · (sine)

3) (polynomial) · (cosine)

In all cases, u = polynomial

Example 4:

$$\int_1^{e^3} (x^7+1)\ln(x)\,dx$$

$\boxed{u = \ln(x)}$ $\quad v = \frac{x^8}{8} + x$

$du = \frac{1}{x}dx \quad dv = \boxed{(x^7+1)dx}$

$$\int_1^{e^3} (x^7+1)\ln(x)\,dx$$

$$= \ln(x)\left(\frac{x^8}{8}+x\right)\Bigg|_1^{e^3} - \int_1^{e^3}\left(\frac{x^8}{8}+x\right)\frac{1}{x}\,dx$$

$$= \ln(x)\left(\frac{x^8}{8}+x\right)\Bigg|_1^{e^3} - \int_1^{e^3}\left(\frac{x^{\cancel{8}\,7}}{8\cancel{x}} + \cancel{\frac{x}{x}}\right)dx$$

$$= \ln(x)\left(\frac{x^8}{8}+x\right)\Big|_1^{e^3} - \underbrace{\int_1^{e^3}\left(\frac{x^7}{8}+1\right)dx}$$

$$\int\left(\frac{x^7}{8}+1\right)dx = \int \frac{x^7}{8}\,dx + \int 1\,dx$$

$$= \frac{1}{8}\int x^7\,dx + \int 1\,dx$$

$$= \frac{1}{8}\cdot\frac{x^8}{8} + x$$

We have

$$\ln(x)\left(\frac{x^8}{8}+x\right)\Big|_1^{e^3} - \left(\frac{x^8}{64}+x\right)\Big|_1^{e^3}$$

$$= \underbrace{\ln(e^3)}_{=3}\left(\frac{e^{24}}{8}+e^3\right) - \left(\frac{e^{24}}{64}+e^3\right) + \left(\frac{1}{64}+1\right)$$

$$= \frac{3e^{24}}{8} + 3e^3 - \frac{e^{24}}{64} - e^3 + \frac{1}{64} + 1$$

$$= \boxed{\frac{3e^{24}}{8} - \frac{e^{24}}{64} + 2e^3 + \frac{1}{64} + 1}$$

Example 5: (exp + trig)

$$\int e^{4x}\sin(x)\,dx$$

$\boxed{u = e^{4x}}$ $\qquad$ $v = -\cos(x)$

$du = 4e^{4x}$ $\qquad$ $\boxed{dv = \sin(x)\,dx}$

$$\int e^{4x}\sin(x)\,dx = -e^{4x}\cos(x) + 4\boxed{\int e^{4x}\cos(x)\,dx}$$

Just as hard as what we started with!

$$\int e^{4x}\cos(x)\,dx$$

Another integration by parts – use the "same" choice for u that you started with.

$$u = e^{4x} \qquad v = \sin(x)$$

$$du = 4e^{4x}\,dx \qquad dv = \cos(x)\,dx$$

$$\int e^{4x}\cos(x)\,dx = e^{4x}\sin(x) - 4\int e^{4x}\sin(x)\,dx$$

So then

$$\int e^{4x}\sin(x)\,dx =$$

$$-e^{4x}\cos(x) + 4\int e^{4x}\cos(x)\,dx$$

$$= -e^{4x}\cos(x) + 4\left(e^{4x}\sin(x) - 4\int e^{4x}\sin(x)\,dx\right)$$

$$= -e^{4x}\cos(x) + 4e^{4x}\sin(x) - 16\int e^{4x}\sin(x)\,dx$$

Add $16\int e^{4x}\sin(x)\,dx$

to both sides.

$$17\int e^{4x}\sin(x)\,dx$$

$$= -e^{4x}\cos(x) + 4e^{4x}\sin(x)$$

Divide by 17.

$$\boxed{\int e^{4x}\sin(x)\,dx = \frac{4e^{4x}\sin(x) - e^{4x}\cos(x)}{17} + C}$$

More Guidelines

1) (polynomial)$\cdot$(logarithm)

   u = logarithm always

2) (exponential)(sine/cosine)

   integrate by parts twice and

   don't switch your choice of u!

Trig Integrals

Section 7.2

Fourier Series: express a function as a "infinite" sum of sines and cosines.

Heat Equation

The differential equation

$$\boxed{f''(x) - cf(x) = 0}$$

where $c < 0$ occurs in the solution for the heat equation.

We can write $0 < -c$. Assume $-c = n^2$ for a counting number n.

We get

$$f''(x) + n^2 f(x) = 0.$$

Let $f(x) = \sin(nx)$

$f'(x) = n\cos(nx)$

$f''(x) = -n^2 \sin(nx).$

Substituting,

$$-n^2 \sin(nx) + n^2(\sin(nx)) = 0.$$

So $f(x) = \sin(nx)$ satisfies this differential equation!

So does $f(x) = \cos(nx)$.

Fourier series allows you to write a function as sines and cosines in a (possibly infinite) sum.

The decomposition requires that we know the following integrals (m, n are counting numbers)

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx$$

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx$$

$$\int_{-\pi}^{\pi} \cos(nx)\sin(mx)\,dx = 0$$

(integral of odd function over symmetric domain).

The easiest nontrivial integral

is $$\int_{-\pi}^{\pi} \sin^2(x)\,dx.$$

Use a trig identity:

$$\boxed{\sin^2(x) = \frac{1-\cos(2x)}{2}}$$

Integral becomes

$$\int_{-\pi}^{\pi} \frac{1-\cos(2x)}{2}\,dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} (1-\cos(2x))\,dx$$

$$\frac{1}{2}\int_{-\pi}^{\pi}(1-\cos(2x))\,dx$$

$$=\frac{1}{2}\left(x-\frac{\sin(2x)}{2}\right)\Big|_{-\pi}^{\pi}$$

$$=\frac{1}{2}\left(\left(\pi-\underbrace{\frac{\sin(2\pi)}{2}}_{=0}\right)-\left(-\pi-\underbrace{\frac{\sin(-2\pi)}{2}}_{=0}\right)\right)$$

$$=\frac{1}{2}\cdot 2\pi=\boxed{\pi}$$

Now for

$$\int_{-\pi}^{\pi} \cos^2(x)\,dx, \text{ use}$$

$$\cos^2(x)+\sin^2(x)=1, \text{ so}$$

$$\cos^2(x)= 1-\sin^2(x).$$

The integral becomes

$$\int_{-\pi}^{\pi} (1-\sin^2(x))\,dx$$

$$= \int_{-\pi}^{\pi} 1\,dx - \int_{-\pi}^{\pi} \sin^2(x)\,dx$$

$$\int_{-\pi}^{\pi} 1\,dx - \int_{-\pi}^{\pi} \sin^2(x)\,dx$$

$$= x \Big|_{-\pi}^{\pi} - \pi$$

$$= 2\pi - \pi = \boxed{\pi}$$

Could also use

$$\boxed{\cos^2(x) = \frac{1+\cos(2x)}{2}}$$